

# MEASURABLE VERSIONS OF THE LS CATEGORY ON LAMINATIONS

CARLOS MENIÑO COTÓN

**ABSTRACT.** We give two new versions of the LS category for the set-up of measurable laminations defined by Bermúdez. Both of these versions must be considered as “tangential categories”. The first one, simply called (LS) category, is the direct analogue for measurable laminations of the tangential category of (topological) laminations introduced by Colman Vale and Macías Virgós. For the measurable lamination that underlies any lamination, our measurable tangential category is a lower bound of the tangential category. The second version, called the  $\Lambda$ -category, depends on the choice of a transverse invariant measure  $\Lambda$ . We show that both of these “tangential categories” satisfy appropriate versions of some well known properties of the classical category: the homotopy invariance, a dimensional upper bound, a cohomological lower bound (cup length), and an upper bound given by the critical points of a smooth function.

## CONTENTS

1. Introduction	1
2. MT-spaces and measurable laminations	2
3. Category of measurable laminations	5
4. $\Lambda$ -category	5
5. Homotopy invariance of the $\Lambda$ -category	8
6. Case of laminations with compact leaves	9
7. Dimensional upper bound	10
8. Cohomological lower bound	14
9. Critical points	17
References	22

## 1. INTRODUCTION

The LS category is a homotopy invariant defined as the minimum number of open subsets contractible within a topological space needed to cover it. Many variations of this invariant has been also defined. In particular, H. Colman Vale and E. Macías Virgós introduced a tangential version for foliations, where they use leafwise contractions to transversals [5, 6].

In this paper, we consider the set-up of measurable laminations defined by Bermúdez [2, 3]. They are “laminations of measurable spaces” in an obvious sense; there is a leaf topology, but the ambient space topology is replaced by a weaker ambient measurable structure. In measurable laminations, we use topological (or

---

Supported by “Ministerio de Ciencia e Innovación”, Spain (grants FPU and MTM2008-02640).

differentiable) terms to refer to the leaf topology, and measurable terms to refer to the ambient measurable structure; for instance, a measurable open set is a set that is leafwise open and measurable in the ambient space. Then we introduce two versions of the tangential category for a measurable lamination  $\mathcal{F}$  on a space  $X$ , which are new even  $\mathcal{F}$  is the underlying measurable lamination of a (topological) lamination. The first one, simply called its (LS) category, is a direct adaptation to measurable laminations of the tangential (or even the usual) category. The second one is called the  $\Lambda$ -category because it involves a transverse invariant measure  $\Lambda$ , whose existence is a restriction.

Let  $\mathcal{F}$  be a measurable lamination on a measurable space  $X$ . The category of  $\mathcal{F}$  is the minimum number of categorical measurable open sets needed to cover  $X$ , where “categorical” means that there exists a measurable continuous deformation to some transversal. Obviously, if  $\mathcal{F}$  is the measurable lamination that underlies any lamination, our category is a lower bound of the tangential category, and sometimes it is easier to study (measurability allows more freedom to make constructions than continuity).

To define the  $\Lambda$ -category of  $\mathcal{F}$  for a transverse invariant measure  $\Lambda$ , we can also take coverings of  $X$  by categorical measurable open sets, but now  $\Lambda$  is used to “count” them: we consider the sum of the measures of the transversals resulting from their deformations. The infimum of all those possible measures is the  $\Lambda$ -category of  $\mathcal{F}$ .

For these two new “tangential categories”, we prove appropriate versions of some classical results about LS category, like their homotopy invariance, a cohomological lower bound given by the cup length, a dimensional upper bound (adapted to the tangential category of foliations by W. Singhof and E. Vogt [22]), or a lower bound given the number of critical points of a smooth function. More precisely, it is proved that the category is less or equal than the number of critical sets of a differentiable function, where the critical sets are defined by using the leafwise gradient flow of the function. This improves even the classical result because a smooth function on a manifold may have less critical sets than critical points. On the other hand, the  $\Lambda$ -category is less or equal than the measure of the set of leafwise critical points; i.e., the critical points of the restrictions of the function to the leaves. Following the work of J. Schwartz [21], these relations with critical points are obtained in the setting of measurable Hilbert laminations because of their possible applications to tangential variational problems [1, 20].

## 2. MT-SPACES AND MEASURABLE LAMINATIONS

A *measurable topological space*, or *MT-space*, is a set  $X$  equipped with a  $\sigma$ -algebra and a topology. Usually, measure theoretic concepts will refer to the  $\sigma$ -algebra of  $X$ , and topological concepts will refer to its topology; in general, the  $\sigma$ -algebra is different from the Borel  $\sigma$ -algebra induced by the topology. An *MT-map* between MT-spaces is a measurable continuous map. An *MT-isomorphism* is a map between MT-spaces that is a measurable isomorphism and a homeomorphism. Trivial examples of MT-spaces are topological spaces with their Borel  $\sigma$ -algebras, and measurable spaces with the discrete topology.

Let  $X$  and  $Y$  be MT-spaces. Suppose that there exists a measurable embedding  $i : X \rightarrow Y$  that maps measurable sets to measurable sets. Then  $X$  is called an *MT-subspace* of  $Y$ . Notice that, if  $X$  and  $Y$  are standard<sup>1</sup>, the measurability of  $i$  means that it maps Borel sets to Borel sets [23]. The product  $X \times Y$  is an MT-space too with the product topology and the  $\sigma$ -algebra generated by products of measurable sets of  $X$  and  $Y$ .

Let  $R$  be an equivalence relation on an MT-space  $X$ . The quotient set  $X/R$  becomes an MT-space with the quotient topology and the  $\sigma$ -algebra generated by the projections of measurable saturated sets of  $X$ ; it can be called the *quotient* MT-space.

Let  $T$  be a standard Borel space and let  $P$  be a Polish space. Let  $P \times T$  be endowed with the structure of MT-space defined by the  $\sigma$ -algebra generated by products of Borel subsets of  $T$  and Borel subsets of  $P$ , and the product of the discrete topology on  $T$  and the topology of  $P$ .

A *measurable chart* on an MT-space  $X$  is an MT-isomorphism  $\varphi : U \rightarrow B \times T$ , where  $U$  is open and measurable in  $X$  and  $B$  is an open ball in  $\mathbb{R}^n$ , the simpler notation  $(U, \varphi)$  may be used for such a chart. The sets  $\varphi^{-1}(B \times \{*\})$  are called *plaques* of  $\varphi$ , and the sets  $\varphi^{-1}(\{*\} \times T)$  are called *transversals* associated to  $\varphi$ . A *measurable atlas* on  $X$  is a countable family of foliated measurable charts whose domains cover  $X$ . A *measurable lamination of dimension  $n$*  is an MT-space that admits a countable measurable atlas consisting of charts  $\varphi_i : U_i \rightarrow B_i \times T_i$  ( $i \in \mathbb{N}$ ), where each  $B_i$  is an open ball in  $\mathbb{R}^n$ . Since we have used a countable atlas, the ambient space is also a standard space. The connected components of  $X$  are called its *leaves*; they are second countable connected manifolds, which may not be Hausdorff. The *saturation* of a set  $B$ , denoted by  $\text{sat}(B)$ , is the union of leaves that meet  $B$ . The saturation of a measurable set is measurable since the measurable atlas is countable.

The typical example of measurable lamination to keep in mind is given by any lamination (or foliation), by considering the Borel  $\sigma$ -algebra of its ambient space and its leaf topology; in this case, “leaves” and “saturations” have the usual meanings.

In the setting of measurable laminations, the concept of  $C^r$  tangential structure cannot be defined as a maximal atlas with (tangentially)  $C^r$  changes of coordinates because the atlases are required to be countable, but we can proceed as follows. A measurable atlas is said to be (tangentially)  $C^r$  if its coordinate changes are (tangentially)  $C^r$ . Then a  $C^r$  structure is an equivalence class of  $C^r$  measurable atlases, where two  $C^r$  measurable atlases are equivalent if their union is a  $C^r$  measurable atlas.

A measurable subset  $T \subset X$  is called a *transversal* if its intersection with each leaf is countable [13]; these are slightly more general than the transversals of [2] that consist of isolated points, this kind of transversals are said to be *isolated*. Let  $\mathcal{T}(X)$  be the family of transversals of  $X$ . This set is closed under countable unions and intersections, but it is not a  $\sigma$ -algebra. A transversal meeting all leaves is called *complete*.

A *measurable holonomy transformation* is a measurable isomorphism  $\gamma : T \rightarrow T'$ , for  $T, T' \in \mathcal{T}(X)$ , which maps each point to a point in the same leaf. A *transverse*

---

<sup>1</sup>Recall that a *Polish space* is a completely metrizable and separable topological space, and a *standard Borel space* is a measurable space isomorphic to a Borel subset of a Polish space.

*invariant measure* on  $X$  is a  $\sigma$ -additive map,  $\Lambda : \mathcal{T}(X) \rightarrow [0, \infty]$ , invariant by measurable holonomy transformations. The classical definition of transverse invariant measure of a lamination is a measure on topological transversals invariant by holonomy transformations (see e.g. [4]); both notions of transverse invariant measures agree in this case [7]. In this paper, we always consider  $\sigma$ -finite measures.

*Remark 1.* There exists an underlying or forgetful functor  $\mathcal{O}$  from the category of foliated spaces and foliated maps (those that map leaves to leaves) to the category of measurable laminations and MT-maps.

The leaf space of a measurable lamination  $\mathcal{F}$  is denoted by  $X/\mathcal{F}$ . Also, for any MT-subspace  $Y \subset X$ ,  $Y/\mathcal{F}$  will denote the quotient space of  $Y$  with the restricted relation (“lying in the same leaf of  $\mathcal{F}$ ”).

Our principal tools in this setting are the following two results (see e.g. [23]).

**Proposition 2.1** (Lusin). *Let  $X$  and  $Y$  be standard Borel spaces and  $f : X \rightarrow Y$  a measurable map with countable fibers. Then  $f(X)$  is Borel in  $Y$  and there exists a measurable section  $s : f(X) \rightarrow X$  of  $f$ . In particular, if  $f$  is injective, then  $s$  is a Borel isomorphism. Moreover there exists a countable Borel partition,  $X = \bigcup_i X_i$ , such that each  $f|_{X_i}$  is injective.*

**Theorem 2.2** (Kunugui, Novikov). *Let  $\{V_n\}_{n \in \mathbb{N}}$  be a countable base for a Polish space  $P$ . Let  $B \subset P \times T$  be a Borel set such that  $B \cap (P \times \{t\})$  is open for every  $t \in T$ . Then there exists a sequence  $\{B_n\}_{n \in \mathbb{N}}$  of Borel sets of  $T$  such that*

$$B = \bigcup_n (V_n \times B_n).$$

Every measurable open MT-subspace  $U$  of a measurable lamination  $\mathcal{F}$  is a measurable lamination (by the previous theorem); the notation  $\mathcal{F}_U$  will be used in this case. By restriction, any transverse invariant measure  $\Lambda$  of  $\mathcal{F}$  induces a transverse invariant measure of  $\mathcal{F}_U$ , which will be denoted by  $\Lambda_U$ .

**Lemma 2.3** ([17]). *Let  $\varphi_i : U_i \rightarrow B_i \times T_i$  and  $\varphi_j : U_j \rightarrow B_j \times T_j$  be measurable charts of  $X$ . There exists a sequence of Borel sets of  $T_i$ ,  $\{S_n\}_{n \in \mathbb{N}}$ , and a base of  $B_i$ ,  $\{V_n\}_{n \in \mathbb{N}}$ , such that  $\varphi_i(U_i \cap U_j) = \bigcup_n (V_n \times S_n)$  and  $\varphi_j \circ \varphi_i^{-1}(x, t) = (g_{ijn}(x, t), f_{ijn}(t))$  for  $(x, t) \in V_n \times S_n$ , where each  $f_{ijn}$  is a Borel isomorphism and each  $g_{ijn}$  is an MT-map.*

*Remark 2.* The previous lemma is also true replacing the open balls  $B_i$  of an euclidean space by any connected and locally connected Polish space.

**Definition 2.4.** A foliated measurable atlas  $\mathcal{U}$  is called *regular* if:

- (i) for all  $(U, \varphi) \in \mathcal{U}$ , there exists another measurable foliated chart  $(W, \psi)$  such that the closure of each plaque in  $U$  is compact,  $\overline{U} \subset W$  and  $\varphi = \psi|_U$ ; and,
- (ii) for all  $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathcal{U}$ , each plaque of  $(U_1, \varphi_1)$  meets at most one plaque of  $(U_2, \varphi_2)$ .

Observe that, if  $\mathcal{U}$  is a regular measurable atlas, then  $\overline{U}$  is measurable for all  $(U, \varphi) \in \mathcal{U}$ .

This definition of regular measurable atlas is weaker than the usual one for laminations (see e.g. [4]): the locally finite condition does not make sense for measurable laminations since there is no ambient topology. The following result follows from Lemma 2.3.

**Corollary 2.5.** *If a measurable lamination has a foliated measurable atlas such that each chart meets a finite number of charts, then it admits a regular measurable foliated atlas.*

From now on, we consider only measurable laminations that admit regular measurable foliated atlases.

**Example 2.6** (Measurable suspensions [2]). Let  $P$  be a connected, locally path connected and semi-locally 1-connected Polish space, and let  $S$  be a standard space. Let  $\text{Meas}(S)$  denote the group of measurable transformations of  $S$ . Let  $h : \pi_1(P, x_0) \rightarrow \text{Meas}(S)$  be a homomorphism. Let  $\tilde{P}$  the universal covering of  $P$  and consider the action of  $\pi_1(P, x_0)$  on the MT-space  $\tilde{P} \times S$  given by  $g \cdot (x, y) = (xg^{-1}, h(g)(y))$ . The corresponding quotient MT-space,  $\tilde{P} \times_h S$ , is called the *measurable suspension* of  $h$ . If  $P$  is a manifold, then  $\tilde{P} \times_h S$  is a measurable lamination,  $\{*\} \times S$  is a complete transversal, and its leaves are covering spaces of  $P$ .

### 3. CATEGORY OF MEASURABLE LAMINATIONS

A measurable lamination  $(X, \mathcal{F})$  induces a foliated measurable structure  $\mathcal{F}_U$  in each measurable open set  $U$  (by Theorem 2.2). The space  $U \times \mathbb{R}$  admits an obvious foliated structure  $\mathcal{F}_{U \times \mathbb{R}}$  whose leaves are products of leaves of  $\mathcal{F}_U$  and  $\mathbb{R}$ . Let  $(Y, \mathcal{G})$  be another measurable lamination. An MT-map  $H : \mathcal{F}_{U \times \mathbb{R}} \rightarrow \mathcal{G}$  is called a (*measurable*) *homotopy*, and it is said that the maps  $H(\cdot, 0)$  and  $H(\cdot, 1)$  are (*measurably*) *homotopic*. We use the term (*measurable*) *deformation* when  $\mathcal{G} = \mathcal{F}$  and  $H(-, 0)$  is the inclusion map of  $U$ . A deformation such that  $H(-, 1)$  is constant on the leaves of  $\mathcal{F}_U$  is called a (*measurable*) *contraction* or an  *$\mathcal{F}$ -contraction*; in this case,  $U$  is called a *categorical* or  *$\mathcal{F}$ -categorical* measurable open set.

The (*LS*) *category* is the lowest number of categorical measurable open sets that cover the measurable lamination. On one leaf foliations, this definition agree with the classical category. The category of  $\mathcal{F}$  is denoted by  $\text{Cat}(\mathcal{F})$ . It is clear that it is a homotopy invariant.

All of these definitions have obvious versions for laminations by using ambient space continuity instead of measurability, obtaining the so called tangential category [5, 6].

**Definition 3.1.** Let  $U \subset (X, \mathcal{F})$  be a measurable open subset. Define the *relative category* of  $U$ ,  $\text{Cat}(U, \mathcal{F})$ , as the minimum number of categorical measurable open sets in  $(X, \mathcal{F})$  that form a cover of  $U$ .

*Remark 3.* Clearly,  $\text{Cat}(U, \mathcal{F}) \leq \text{Cat}(\mathcal{F}_U)$ .

**Proposition 3.2** (Subadditivity of the relative category). *Let  $\{U_i\}_{i \in \mathbb{N}}$  be a countable family of measurable open subsets of  $X$ . Then*

$$\text{Cat}\left(\bigcup_i U_i, \mathcal{F}\right) \leq \sum_i \text{Cat}(U_i, \mathcal{F}).$$

### 4. $\Lambda$ -CATEGORY

**Lemma 4.1** (Kallman [15]). *Let  $P \times T$  be a product of a Polish space  $P$  with a standard space  $T$ , and let  $\pi : P \times T \rightarrow T$  denote the second factor projection. Let  $B \subset P \times T$  be a measurable subset such that  $B \cap (P \times \{t\})$  is  $\sigma$ -compact for all  $t \in T$ . Then  $\pi(B)$  is measurable.*

**Proposition 4.2.** *For any measurable open set  $U$  and any tangential deformation  $H$ , the set  $H(U \times \{1\})$  is measurable.*

*Proof.* By the Kunugui-Novikov's theorem (Theorem 2.2), there exist countable families,  $\{P_n \times T_n\}$  and  $\{P'_n \times T'_n\}$  ( $n \in \mathbb{N}$ ), and MT-embeddings  $f_n : P_n \times T_n \rightarrow \mathcal{F}$  and  $g_n : P'_n \times T'_n \rightarrow \mathcal{F}$  such that  $U = \bigcup_n f_n(P_n \times T_n)$  and  $H(\cdot, 1) \circ f_n(P_n \times T_n) \subset g_n(P'_n \times T'_n)$ . Hence we only have to prove that each  $g_n^{-1} \circ H(\cdot, 1) \circ f_n(P_n \times T_n)$  is measurable. Consider a topology on  $T_n$  so that it is isomorphic to  $[0, 1]$ ,  $\mathbb{N}$  or a finite set [16], and let  $P_n \times T_n$  be endowed with the product topology  $\tau$ , becoming a Polish space. Let  $\pi : (P_n \times T_n, \tau) \times P'_n \times T'_n \rightarrow P'_n \times T'_n$  be the second factor projection. For each point  $(x, t) \in P'_n \times T'_n$ , the set  $f_n^{-1} \circ H(\cdot, 1)^{-1} \circ g_n(x, t)$  is  $\sigma$ -compact in  $(P_n \times T_n, \tau)$ . By the continuity of  $H(\cdot, 1) \circ f_n$  (with the leaf topology), this preimage is closed on each plaque  $P_n \times \{t\}$  ( $t \in T_n$ ). On the other hand, this preimage only cuts a countable number of leaves (otherwise  $H$  would not be a tangential deformation), and therefore it only cuts a countable number of plaques. Hence this preimage is also  $\sigma$ -compact with the topology  $\tau$ . Let

$$B_n = \{((x, t), (x', t')) \mid g_n^{-1} \circ H(\cdot, 1) \circ f_n(x, t) = (x', t')\},$$

which is measurable in  $(P_n \times T_n, \tau) \times P'_n \times T'_n$ . Then, by Lemma 4.1,  $\pi(B_n)$  is measurable. But  $\pi(B_n) = g_n^{-1} \circ H(\cdot, 1) \circ f_n(P_n \times T_n)$ .  $\square$

**Proposition 4.3** ([17]). *Let  $(X, \mathcal{F})$  be a measurable lamination with a transverse invariant measure  $\Lambda$ . There exists a unique Borel measure  $\tilde{\Lambda}$  on  $X$  such that  $\tilde{\Lambda}(T) = \Lambda(T)$  for all generalized transversal  $T$ , and satisfying the following properties:*

- (i) *If  $B$  is a measurable set so that  $B \not\subset \pi^{-1}(S)$  for any transversal  $S$  with  $\Lambda(S) = 0$ , and  $\partial(B \cap L) \neq \emptyset$  for each leaf  $L$  that meets  $B$ , then  $\tilde{\Lambda}(B) = \infty$ .*
- (ii) *If  $\Lambda(S) = 0$  for some  $S \in \mathcal{B}$ , then  $\tilde{\Lambda}(\text{sat}(S)) = 0$ .*
- (iii) *If  $B$  is a measurable set so that  $\Lambda(S) = \infty$  for all transversal  $S$  with  $B \subset \text{sat}(S)$ , then  $\tilde{\Lambda}(B) = \infty$ .*

The unique extended measure  $\tilde{\Lambda}$ , given by Proposition 4.3, is called the *coherent extension* of  $\Lambda$ .

*Remark 4.* Observe that, in measurable charts of the form  $P \times T$ , the coherent extension can be given on measurable sets with  $\sigma$ -compact intersection with the plaques as the pairing of the counting measure with the transverse invariant measure  $\Lambda$  [17]:

$$\tilde{\Lambda}(B) = \int_T \#(B \cap (P \times \{t\})) d\Lambda(t).$$

Let  $\Lambda$  be a transverse invariant measure for  $\mathcal{F}$ , and  $\tilde{\Lambda}$  its coherent extension. According to Proposition 4.2, we can define

$$\tau_\Lambda(U) = \inf\{\tilde{\Lambda}(H(U \times \{1\})) \mid H \text{ is a tangential deformation of } U\}.$$

Then the  $\Lambda$ -category of  $(\mathcal{F}, \Lambda)$  is defined as

$$\text{Cat}(\mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} \tau_\Lambda(U),$$

where  $\mathcal{U}$  runs in the countable coverings of  $X$  by measurable open sets. The countability condition of the coverings is needed, otherwise the  $\Lambda$ -category would be directly zero when  $\Lambda$  has no atoms. If the homotopies used in this definition are

required to be  $C^r$  on leaves, then the term  $C^r$   $\Lambda$ -category is used, with the notation  $\text{Cat}^r(\mathcal{F}, \Lambda)$ .

The category and the  $\Lambda$ -category are connected by the following result.

**Proposition 4.4.** *Let  $(X, \mathcal{F}, \Lambda)$  be a measurable lamination with a transverse invariant measure, and let  $U$  be a measurable open set in  $X$ . If  $\tau_\Lambda(U) < \infty$ , then there exists a categorical measurable open set  $U' \subset U$  such that  $\tilde{\Lambda}(U \setminus U') = 0$ .*

*Proof.* There exists a tangential deformation of  $U$  such that  $\tilde{\Lambda}(H(U \times \{1\})) < \infty$ . This means, by the conditions of the coherent extension, that  $H(U \times \{1\}) = B \cup T$ , where  $B$  is a Borel set with  $\tilde{\Lambda}(B) = 0$  and  $T$  is a transversal of  $\mathcal{F}$ . The set  $U' = H(\cdot, 1)^{-1}(T)$  satisfies the required conditions.  $\square$

**Definition 4.5.** Let  $(X, \mathcal{F}, \Lambda)$  be a foliated measurable space with a transverse invariant measure. A *null-transverse* set is a measurable set  $B$  such that  $\tilde{\Lambda}(B) = 0$ .

The following propositions are elementary.

**Proposition 4.6.** *Let  $(X, \mathcal{F}, \Lambda)$  be a measurable lamination with a transverse invariant measure, and let  $B$  be a null-transverse set. Then  $\text{Cat}(\mathcal{F}, \Lambda)$  can be computed by using only coverings of  $X \setminus \text{sat}(B)$ . If  $B$  is saturated, then  $\text{Cat}(\mathcal{F}, \Lambda) = \text{Cat}(\mathcal{F}_{X \setminus B}, \Lambda_{X \setminus B})$ .*

**Proposition 4.7.** *Let  $T$  be a measurable transversal which meets each leaf at most in one point. Then  $\Lambda(T) \leq \text{Cat}(\mathcal{F}, \Lambda)$ .*

**Proposition 4.8.** *Let  $(X, \mathcal{F}, \Lambda)$  be a measurable lamination with a transverse invariant measure. If  $(X, \mathcal{F})$  is a measurable suspension  $\tilde{M} \times_h S$ , then  $\text{Cat}(\mathcal{F}, \Lambda) \leq \text{Cat}(M) \cdot \Lambda(S)$ .*

**Proposition 4.9.** *For a manifold  $M$  and a standard Borel space  $T$ , let  $M \times T$  be foliated as a product. Then  $\text{Cat}(M \times T, \Lambda) = \text{Cat}(M) \cdot \Lambda(T)$  for every measure  $\Lambda$  on  $T$ , considered as an invariant measure of  $M \times T$ .*

**Proposition 4.10.** *Let  $\{U_n\}$  ( $n \in \mathbb{N}$ ) be a covering by saturated measurable open sets of  $(X, \mathcal{F}, \Lambda)$ . Then  $\text{Cat}(\mathcal{F}, \Lambda) \leq \sum_n \text{Cat}(\mathcal{F}_{U_n}, \Lambda_{U_n})$ . Here, the equality holds if  $\{U_n\}$  is a partition.*

**Definition 4.11.** Let  $U \subset (X, \mathcal{F})$  be a measurable open subset. The *relative  $\Lambda$ -category* of  $U$  is defined by

$$\text{Cat}(U, \mathcal{F}, \Lambda) = \inf_{\mathcal{U}} \sum_{V \in \mathcal{U}} \tau_\Lambda(V),$$

where  $\mathcal{U}$  runs in the family of countable measurable open coverings of  $U$ .

*Remark 5.* Let us emphasize that, in the above definition,  $\tau_\Lambda(V)$  is defined by using measurable tangential homotopies deforming  $V$  in the ambient space. Clearly,  $\text{Cat}(U, \mathcal{F}, \Lambda) \leq \text{Cat}(\mathcal{F}_U, \Lambda)$ .

**Proposition 4.12** (Subadditivity of the relative  $\Lambda$ -category). *Let  $\{U_i\}_{i \in \mathbb{N}}$  be a countable family of measurable open subsets of  $X$ . Then*

$$\text{Cat}\left(\bigcup_i U_i, \mathcal{F}, \Lambda\right) \leq \sum_i \text{Cat}(U_i, \mathcal{F}, \Lambda).$$

5. HOMOTOPY INVARIANCE OF THE  $\Lambda$ -CATEGORY

Let  $\mathcal{F}$  and  $\mathcal{G}$  be measurable laminations. An MT-homotopy equivalence  $h$  from  $\mathcal{F}$  to  $\mathcal{G}$  induces a canonical bijection between the sets of transverse invariant measures on  $\mathcal{G}$  and  $\mathcal{F}$ , which is defined as follows. Let  $T$  be a complete transversal of  $\mathcal{F}$ . Obviously  $h|_T$  has countable fibers. By Proposition 2.1,  $h(T)$  is a transversal of  $\mathcal{G}$ . In fact, there exists a countable measurable partition,  $T = \bigcup_i T_i$ , so that each  $h|_{T_i}$  is injective. Define  $h^*\Lambda(T) = \sum_i \Lambda(h(T_i))$ . From now on, we any MT-homotopy equivalence between measurable laminations with transverse invariant measures is assumed to be compatible with the measures in the above sense.

**Lemma 5.1.** *Let  $(X, \mathcal{F}, \Lambda)$  and  $(Y, \mathcal{G}, \Delta)$  be foliated measurable spaces with transverse invariant measures, and let  $h : (X, \Lambda) \rightarrow (Y, \Delta)$  be a measurable homotopy equivalence. Then, for all measurable set  $K \subset X$  with  $\sigma$ -compact intersections with the leaves,  $h(K)$  is measurable and  $\tilde{\Delta}(h(K)) \leq \tilde{\Lambda}(K)$ .*

*Proof.* The fact that  $h(K)$  is measurable is a consequence of Proposition 4.2. Notice that finite intersections of  $\sigma$ -compact sets are  $\sigma$ -compact. Let  $\mathcal{U} = \{(U_n, \varphi_n)\}$  and  $\mathcal{V} = \{(V_n, \psi_n)\}$  ( $n \in \mathbb{N}$ ) be foliated measurable atlases for  $\mathcal{F}$  and  $\mathcal{G}$ , respectively. Observe that there exists a foliated measurable atlas  $\mathcal{W} = \{(W_n, \phi_n)\}$  on  $X$  satisfying the following conditions:

- (a) For each  $n \in \mathbb{N}$ , there exists some  $k(n) \in \mathbb{N}$  such that  $h(W_n) \subset V_{k(n)}$ .
- (b) The map  $h$  induces an injective map from the family of plaques of  $W_n$  to the family of plaques of  $V_{k(n)}$ ; i.e., each plaque of  $V_{k(n)}$  contains at most the image by  $h$  of one plaque of  $W_n$ .

This atlas can be easily obtained by using Theorem 2.2 and Proposition 2.1.

The maps  $\psi_{k(m)} \circ h \circ \phi_m^{-1} : P_m \times T_m \rightarrow P'_{k(m)} \times T'_{k(m)}$  are injective in the set of plaques in the sense of (b). Clearly,  $\Lambda(D) = \Delta(h(D))$  for all Borel sets  $D \subset T_m$ . For every  $t \in T_m$ , let  $P^t$  be the plaque of  $P'_{k(m)} \times T'_{k(m)}$  that contains  $\psi_{k(m)} \circ h \circ \phi_m^{-1}(P_m \times \{t\})$ . We have

$$\#(h(K) \cap P^t) \leq \#(K \cap (P \times \{t\}))$$

for every  $t \in T_m$ . Hence

$$\begin{aligned} \tilde{\Delta}(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K)) &= \int_{T'_{k(m)}} \#(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K) \cap (P' \times \{t'\})) d\Delta(t') \\ &= \int_{\psi_{k(m)} \circ h \circ \phi_m^{-1}(T_m)} \#(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K) \cap (P \times \{t'\})) d\Delta(t') \\ &= \int_{T_m} \#(\psi_{k(m)} \circ h \circ \phi_m^{-1}(K) \cap P^t) d\Lambda(t) \\ &\leq \int_{T_m} \#(K \cap (P \times \{t\})) d\Lambda(t) \\ &= \tilde{\Lambda}(K). \end{aligned}$$

Then the lemma holds on each chart of  $\mathcal{W}$ . Consider the family  $\{B_k\}$  inductively defined by

$$B_1 = (K \cap W_1), \quad B_k = (K \cap W_k) \setminus (B_1 \cup \dots \cup B_{k-1}) \quad (k > 1).$$



It is a Borel partition of  $K$  whose elements are contained in the sets  $W_k$  and have  $\sigma$ -compact intersection with each leaf. Then

$$\tilde{\Delta}(h(K)) = \tilde{\Delta}\left(\bigcup_i h(B_i)\right) \leq \sum_i \tilde{\Delta}(h(B_i)) \leq \sum_i \tilde{\Lambda}(B_i) = \tilde{\Lambda}(K). \quad \square$$

**Proposition 5.2** (The  $\Lambda$ -category is an MT-homotopy invariant). *Let  $(X, \mathcal{F}, \Lambda)$  and  $(Y, \mathcal{G}, \Delta)$  be measurable homotopy equivalent measurable laminations with transverse invariant measures. Then  $\text{Cat}(\mathcal{F}, \Lambda) = \text{Cat}(\mathcal{G}, \Delta)$ .*

*Proof.* Let  $h : X \rightarrow Y$  be a measurable homotopy equivalence, and let  $g$  be a homotopy inverse of  $h$ . Let  $\{U_n\}$  ( $n \in \mathbb{N}$ ) be a covering of  $Y$  by measurable open sets. Then  $\{h^{-1}(U_n)\}$  is a covering of  $X$  by measurable open sets. We will prove that  $\tau_\Lambda(h^{-1}(U_n)) \leq \tau_\Delta(U_n)$  for all  $n \in \mathbb{N}$ . Let  $H^n$  be a measurable tangential deformation on each  $U_n$ , and let  $F$  be an MT-homotopy connecting the identity map and  $g \circ h$ . Let

$$G = g \circ H \circ (f \times \text{id}) : h^{-1}(U) \times \mathbb{R} \rightarrow X.$$

Then  $K : h^{-1}(U) \times \mathbb{R} \rightarrow X$ , defined by

$$K(x, t) = \begin{cases} F(x, 2t) & \text{if } t \leq 1/2 \\ G(x, 2t - 1) & \text{if } t \geq 1/2, \end{cases}$$

is a tangential deformation. Lemma 5.1 yields

$$\tilde{\Lambda}(K(h^{-1}(U) \times \{1\})) = \tilde{\Lambda}(g(H(U \times \{1\}))) \leq \tilde{\Delta}(H(U \times \{1\})).$$

Hence  $\tau_\Lambda(h^{-1}(U_n)) \leq \tau_\Delta(U_n)$  for all  $n \in \mathbb{N}$ . Therefore  $\text{Cat}(\mathcal{F}, \Lambda) \leq \text{Cat}(\mathcal{G}, \Delta)$ . The inverse inequality is analogous.  $\square$

The above proposition has an obvious  $C^r$  version.

## 6. CASE OF LAMINATIONS WITH COMPACT LEAVES

In this section, we compute the  $\Lambda$ -category of (the measurable laminations underlying) a lamination  $\mathcal{F}$  with compact leaves on a Polish space  $X$ . With these conditions, there exists a countable filtration

$$\cdots \subset E_\alpha \subset \cdots \subset E_2 \subset E_1 \subset E_0 = X,$$

such that each  $E_\alpha$  is a closed saturated set, and  $E_\alpha \setminus E_{\alpha+1}$  is dense in  $E_\alpha$  and consists of leaves with trivial holonomy on the foliated space  $E_\alpha$ . This family is called the *Epstein filtration* of  $X$  [10, 11, ?].

Obviously, each  $E_\alpha \setminus E_{\alpha+1}$  is a saturated measurable open set (in the MT-structure) without holonomy. Hence, by Proposition 4.10,

$$(6.1) \quad \text{Cat}(\mathcal{O}(\mathcal{F}), \Lambda) = \sum_{\alpha} \text{Cat}(\mathcal{O}(\mathcal{F}_{E_{\alpha-1} \setminus E_\alpha}), \Lambda_{E_{\alpha-1} \setminus E_\alpha}),$$

where  $\mathcal{O}$  denotes the underlying functor defined in the Remark 1.

**Theorem 6.1** (See e.g. [24]). *Let  $R$  be an equivalence relation on a Polish space  $X$  such that every equivalence class is a closed set in  $X$ . If the saturations of open sets of  $X$  are Borel, then there exists a Borel set meeting every equivalence class in one point. If the saturations of open sets are open, then there exists a Polish subspace meeting every equivalence class in one point.*

**Corollary 6.2.** *Let  $(X, \mathcal{F})$  be a lamination with all leaves compact and let  $T$  be a complete transversal of  $\mathcal{F}$ . Then there exists a Polish subspace contained in  $T$  meeting every leaf in one point.*

Let  $\Gamma$  be the holonomy pseudogroup of  $\mathcal{F}$  on  $T$ . Let  $Q \subset T$  be a Polish subspace satisfying the statement of Theorem 6.1 for the equivalence relation defined by the orbits of  $\Gamma$  on  $T$ . There exists a bijective map  $\pi : Q \rightarrow T/\Gamma \equiv X/\mathcal{F}$  induced by the projection to the leaf space. This map is measurable with respect to the Borel  $\sigma$ -algebra of  $X/\mathcal{F}$ . By using the Epstein filtration of  $(X, \mathcal{F})$ , it is easy to see that  $\pi$  is a Borel isomorphism, even when  $X/\mathcal{F}$  is not Hausdorff. If  $\Lambda$  is a transverse invariant measure, then it induces a measure on  $Q$  since it is a Borel transversal. Hence  $\Lambda$  induces a measure  $\Lambda_{\mathcal{F}}$  on  $X/\mathcal{F}$  via the Borel isomorphism  $\pi$ . The measure  $\Lambda_{\mathcal{F}}$  is independent of the choice of the Polish (Borel) sets  $R$  and  $T$ , since all of them are equivalent by a measurable holonomy map. By using the Epstein filtration, it easily follows that the *category map*,  $\text{Cat} : X/\mathcal{F} \rightarrow \mathbb{N} \cup \{\infty\}$ , assigning to each leaf its category, is measurable.

By (6.1), we can assume that the leaves have trivial holonomy groups to compute  $\text{Cat}(\mathcal{O}(\mathcal{F}), \Lambda)$ . Moreover  $X$  has a countable number of connected components, and the leaves on each connected component are homeomorphic since the holonomy groups are trivial. So, in general,  $\mathcal{O}(\mathcal{F})$  is MT-isomorphic to the disjoint union of product spaces  $L_n \times R_n$ , where  $L_n$  are the generic leaves in each connected component and  $R_n$  are Borel sets meeting leaves in one point in these components. By Propositions 4.9 and 4.10, we obtain the following corollary.

**Corollary 6.3.** *Let  $(X, \mathcal{F}, \Lambda)$  be a foliated space with compact leaves endowed with a transverse invariant measure. Then*

$$\text{Cat}(\mathcal{O}(\mathcal{F}), \Lambda) = \int_{X/\mathcal{F}} \text{Cat}(L) d\Lambda_{\mathcal{F}}(L) .$$

*Remark 6.* By Proposition 4.6, this Corollary 6.3 applies to the case of a measurable lamination with a transverse invariant measure supported on a countable number of compact leaves.

By using the same decomposition, it is easy to check that the measurable tangential category is exactly the maximum of the LS-category of the leaves,  $\text{Cat}(\mathcal{O}(\mathcal{F})) = \max\{\text{Cat}(L) \mid L \in \mathcal{F}\}$ .

## 7. DIMENSIONAL UPPER BOUND

It is known that  $\text{Cat}(M) \leq \dim M + 1$  for any manifold  $M$  [14]. This result was adapted to the tangential category of a  $C^2$  foliation  $\mathcal{F}$  on closed manifold by E. Vogt and W. Singhoff [22], obtaining  $\text{Cat}(\mathcal{F}) \leq \dim \mathcal{F} + 1$ . We show an adaptation to our measurable setting.

Suppose that there exists a complete Riemannian metric  $g$  on the leaves of a  $C^1$  measurable lamination which varies in a measurable way on the ambient space. Clearly, the exponential map is measurable by the assumptions on  $g$ .

**Lemma 7.1.** *The function  $i : X \rightarrow \mathbb{R}^+ \cup \{\infty\}$ , defined as the injectivity radius of the exponential map at each point, is measurable.*

*Proof.* Let  $\mathcal{U} = \{U_l\}_{l \in \mathbb{N}}$  be a regular measurable atlas, where  $U_l \cong B_l \times T_l$ . Clearly,  $i$  is measurable on the leaves since the injectivity radius is a lower semicontinuous

map. Consider the Borel  $\sigma$ -algebra associated to the compact-open topology in  $C(B_l, \mathbb{R}^{m^2})$ . Then the Riemannian metric  $g$  on the chart  $U_l$  can be considered as a measurable map  $g : T \rightarrow C(B_l, \mathbb{R}^{m^2})$ , where  $g(t)(x)$  is the matrix of coefficients of  $g$  at  $(x, t)$  with respect to the canonical frame of  $T\mathbb{R}^m$ . In fact, we can work with the closure  $\overline{B_l} \times T$ . Let  $\{\{U(p_i^n)\}_{i \in \mathbb{N}}\}_{n \in \mathbb{N}}$  be a sequence of open coverings of  $C(\overline{B_l}, \mathbb{R}^{m^2})$ , where  $p_i^n \in C(\overline{B_l}, \mathbb{R}^{m^2})$ , and  $U(p_i^n)$  consists of the functions  $f \in C(\overline{B_l}, \mathbb{R}^{m^2})$  such that  $\|f - p_i^n\| < 2^{-n}$ , using the norm of the maximum absolute value of the coefficients in  $\mathbb{R}^{m^2}$ . Therefore  $T_i^n = \{g^{-1}(U(p_i^n))\}_{i \in \mathbb{N}}$  is a covering of  $T$  by measurable sets. By definition, for  $t, t' \in T_i^n$ ,  $\|g(x, t) - g(x, t')\| < 2^{-n}$  for all  $x \in \overline{B_l}$ .

Let  $U_{l_1}, \dots, U_{l_N}$  be a finite sequence of measurable charts (it is possible that  $U_{l_i} = U_{l_j}$  for some  $i \neq j$ ) and let  $U = \bigcup_{j=1}^N U_{l_j}$ . Then  $U$  can be decomposed into a countable family of product foliations  $\mathcal{F}_n$  such that  $\mathcal{F}_n \cap U_{l_j}$  is saturated in  $U_{l_j}$  for each  $j$ . Of course, we can do the previous argument on each product foliation of the given decomposition.

Since the family of finite collections of measurable charts is countable, we have proved that the lamination  $\mathcal{F}$  is a countable union of products  $\{K_i^n \times T_i^n\}_{i \in \mathbb{N}}$ , where each  $K_i^n$  is compact,  $\|g(x, t) - g(x, t')\| < 2^{-n}$  for all  $x \in K_i^n$  and  $t, t' \in T_i^n$ , and, for each  $x \in \mathcal{F}$ , there exists an expansive sequence  $K_{i_x}^n \subset K_{i_x}^{n+1} \subset \dots$  meeting  $x$  such that  $\bigcup_{n \in \mathbb{N}} K_{i_x}^n = L_x$ , where  $L_x$  is the leaf through  $x$ . This final property is a consequence of the fact that these  $K_i^n$  are finite unions of closures of plaques in chains of charts associated to each finite sequence of charts in  $\mathcal{U}$ .

Finally, the injectivity radius map is measurable by the lower semicontinuity relative to the variation of the metric. For  $n$  big enough, the Riemannian metric on the plaques of the products  $K_i^n \times T_i^n$  is so close to each other as we want. Let  $x \in X$  be a point where the injectivity radius is greater than  $r \in \mathbb{R}$ . By the lower semicontinuity, choose  $n$  such that  $x \simeq (x_0, t_0) \in \text{int}(K_i^n \times T_i^n)$  and the injectivity radius of each point  $(y, t)$ ,  $(y, t) \in B_x \times T_i^n$ , is also greater than  $r$ ; where  $B_x \subset K_i^n$  is an open neighborhood of  $x_0$ . Being the sequences of products  $\{K_i^n \times T_i^n\}_{n, i \in \mathbb{N}}$  a countable collection and leaves second countable, we deduce that  $i^{-1}(r, \infty]$  is measurable for all  $r \in \mathbb{R}$ .  $\square$

**Definition 7.2** (Measurable triangulation [2]). Let  $\Delta^n$  denote the canonical  $n$ -simplex. A *measurable triangulation* is a family of triangulations on the leaves,  $\{\mathcal{T}_L\}_{L \in \mathcal{F}}$ , which is *measurable* in the following sense:

- the sets of barycenters of  $n$ -simplices,  $\mathcal{B}^n$ , are transversals of  $\mathcal{F}$ ; and
- the maps  $\sigma^n : \Delta^n \times \mathcal{B}^n \rightarrow X$  are measurable, where  $\sigma^n(p, \cdot) : \Delta^n \rightarrow L_p$  is the simplex of  $\mathcal{T}_{L_p}$  with barycenter  $p$ .

A measurable triangulation is of class  $C^m$  if the functions  $\sigma_p^n$  are  $C^m$ .

We work in this section with  $C^1$  measurable triangulations.

Let  $\mathcal{T}$  be a measurable triangulation. Let  $\mathcal{T}^n$  denote the family of  $n$ -faces of  $\mathcal{T}$  (the  $n$ -simplices of  $\mathcal{T}$  without their boundaries).

**Proposition 7.3.** *Let  $T$  be an isolated transversal. There exists a measurable open neighborhood  $U(T)$  of  $T$  such that the closures of its connected components are disjoint and contain only one point of  $T$ . In fact,  $U(T)$  can be contracted to  $T$  in a measurable way.*

*Proof.* Since  $T$  is isolated and Borel, the function  $h : T \rightarrow \mathbb{R} \cup \{\infty\}$ , defined by

$$h(p) = \inf\{d_g(p, p') \mid p' \in (T \cap L_p) \setminus \{p\}\},$$

where  $L_p$  denotes the leaf meeting  $p$  and  $d_g$  is the distance map on the leaves induced by the metric  $g$ , is measurable (by the measurability of  $d_g$ ) and positive. The set  $h^{-1}(\infty)$  is measurable. Redefine  $h$  in this set to be identically equal to 1, and hence  $0 < h < \infty$ . Now, let

$$U(T) = \bigcup_{p \in T} B^g(p, \min\{h(p), i(p)\}/2),$$

where  $B^g(p, \varepsilon)$  is the  $d_g$ -ball in  $L_p$  with center on  $p$  and radius  $\varepsilon$ . This set is measurable since  $h$  and  $i$  are measurable. Obviously, the connected components are the balls  $B^g(p, \min\{h(p), i(p)\}/2)$  and satisfy the required conditions. A measurable contraction to  $T$  is given by the radial contraction on the tangent space via the exponential map.  $\square$

**Definition 7.4.** Let  $H : U \times \mathbb{R} \rightarrow X$  and  $G : V \times \mathbb{R} \rightarrow X$  be tangential deformations such that  $H(U \times \{1\}) \subset V$ . Let  $H * G : U \times \mathbb{R} \rightarrow X$  be the tangential deformation defined by

$$H * G(x, t) = \begin{cases} H(x, 2t) & \text{if } t \leq \frac{1}{2} \\ G(H(x, 1), 2t - 1) & \text{if } \frac{1}{2} \leq t. \end{cases}$$

**Lemma 7.5.** Let  $T$  be a standard Borel space, let  $\mathbb{R}^m \times T$  be endowed with the usual MT-structure and let  $\pi : \mathbb{R}^m \times T \rightarrow T$  be the canonical projection. Let  $S$  be a transversal that meets each plaque of  $\mathbb{R}^m \times T$  at most in one point. Then there exists a measurable homotopy  $H : S \times \mathbb{R} \rightarrow \mathbb{R}^m \times T$  such that  $H(s, 0) = s$  and  $H(s, 1) = (0, \pi(s))$ .

*Proof.* Consider the measurable homotopy

$$G : (\mathbb{R}^m \times T) \times \mathbb{R} \rightarrow \mathbb{R}^m \times T, \quad ((v, t), s) \mapsto ((1 - s)v, t).$$

Then  $H = G|_{S \times \mathbb{R}}$  satisfies the conditions of the statement.  $\square$

**Corollary 7.6.** Let  $T$  and  $T'$  be transversals in a measurable chart  $U$  which are bijective via the canonical projection map. Then there exists a measurable homotopy  $H : T \times \mathbb{R} \rightarrow U$  such that  $H(t, 0) = t$  and  $H(t, 1) \in T' \cap P_t$  for all  $t \in T$ , where  $P_t$  is the plaque containing  $t$ .

**Definition 7.7.** A chain of charts of a measurable foliated atlas  $\mathcal{U} = \{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$  is a finite sequence,  $\mathcal{C} = (U_{i_0}, \dots, U_{i_n})$ , such that  $U_{i_j} \cap U_{i_{j+1}} \neq \emptyset$  for all  $j$ . The chain of charts  $\mathcal{C}$  covers a path  $c : I = [0, 1] \rightarrow X$  if there exists a partition  $0 = t_0 < t_1 < \dots < t_n = 1$  of  $I$  such that  $c([t_{j-1}, t_j]) \subset U_{i_j}$  for all  $j$ . The length of a chain of charts  $\mathcal{C} = \{U_{i_0}, \dots, U_{i_n}\}$  is  $n$ . In a similar way we can define a chain of plaques and a the length of a chain of plaques.

Let  $c : I \rightarrow X$  be a foliated path (i.e., a path contained in one leaf). Any chain of charts  $\mathcal{C} = (U_{i_0}, \dots, U_{i_n})$  covering  $c$  induces a measurable holonomy map  $h_{\mathcal{C}}$  between transversals containing  $c(0)$  and  $c(1)$  like in the topological case.

**Lemma 7.8.** Let  $(X, \mathcal{F})$  be a measurable lamination that admits a regular foliated measurable atlas. Let  $c : I \rightarrow X$  be a foliated path. Let  $\mathcal{C} = (U_{i_0}, \dots, U_{i_n})$  be a chain of charts covering  $c$ , and  $h_{\mathcal{C}}$  the measurable holonomy map induced by  $\mathcal{C}$ . Let

$T$  be the domain of  $h_C$ . Then there exists a measurable homotopy  $H : T \times \mathbb{R} \rightarrow X$  such that  $H(t, 0) = t$  and  $H(t, 1) = h_C(t)$ .

*Proof.* There exist  $n - 1$  transversals,  $S_1, \dots, S_{n-1}$ , such that  $S_k \subset U_{i_k} \cap U_{i_{k+1}}$ , which meet only plaques that cut these intersections, and only at one point (by Theorem 2.2). By Corollary 7.6, we obtain the required homotopy.  $\square$

**Lemma 7.9.** *Suppose that  $\mathcal{F}$  admits a regular foliated measurable atlas. Let  $h : T \rightarrow T'$  be a measurable holonomy map. Then there exists a measurable homotopy  $H : T \times \mathbb{R} \rightarrow X$  such that  $H(t, 0) = t$  and  $H(t, 1) = h(t)$ .*

*Proof.* By Proposition 2.1, we can suppose that  $T$  and  $T'$  are contained in transversals associated to charts in the regular foliated measurable atlas. Observe that there exists a countable number of chains of charts covering foliated paths connecting points of  $T$  and  $T'$ ; the induced measurable holonomy maps are denoted by  $\{h_1, h_2, \dots\}$ . The sets

$$B_n = \{t \in T \mid h_n(t) = h(t)\}$$

are measurable. By induction, define

$$C_1 = B_1, \quad C_n = B_n \setminus (C_1 \cup \dots \cup C_{n-1}) \quad (n > 1).$$

These transversals form a partition of  $T$  and, by Lemma 7.8, there exist homotopies  $H^i : C_i \times \mathbb{R} \rightarrow X$  such that  $H^i(t, 0) = t$  and  $H^i(t, 1) = h_i(t) = h(t)$ . The Borel sets  $H^i(C_i \times \{1\})$  form a partition of  $T'$  since  $h$  is a bijection. Combining these homotopies, we obtain the desired homotopy.  $\square$

**Proposition 7.10.** *Let  $U$  be a categorical open set, and let  $T$  be a complete transversal. There exists a measurable contraction  $H$  of  $U$  so that  $H(U \times \{1\}) \subset T$ .*

*Proof.* Let  $F$  be a contractible homotopy for  $U$ . Therefore  $T_F = F(U \times \{1\})$  is a measurable transversal. Since  $T$  is a complete transversal, by Proposition 2.1, there exists a countable partition of  $T_F$  into measurable transversals  $\{T_F^i\}$  ( $i \in \mathbb{N}$ ), and there are injective measurable holonomy maps  $h_i : T_F^i \rightarrow T$ . By the above arguments, these holonomy maps induce a measurable homotopy  $G : T_F \times \mathbb{R} \rightarrow \mathcal{F}$  such that  $G(x, 0) = x$  and  $G(T_F \times \{1\}) \subset T$ . Then  $H = F * G$  is the required contraction.  $\square$

By Proposition 7.3,  $\mathcal{T}^0$  is contained in a categorical open set. Now, we prove an analogous property for each  $\mathcal{T}^n$  for  $0 < n \leq \dim \mathcal{F}$ .

**Proposition 7.11.** *There exists a measurable triangulation  $\mathcal{T}'$  and categorical open sets  $U^n$  such that  $\mathcal{T}'^n \subset U^n$  for  $0 < n \leq \dim \mathcal{F}$ .*

*Proof.* Let  $e(x)$  be the  $n$ -face ( $n$ -simplex without boundary) containing  $x$ . Using barycentric division, we can suppose that all  $n$ -faces,  $0 < n \leq \dim \mathcal{F}$ , are contained in an exponential ball centered at the corresponding barycenter (a geodesic ball with radius smaller than the injectivity radius); this triangulation will be called  $\mathcal{T}'$ . In fact, we can suppose that the diameter of  $e(p)$  is smaller than  $i(p)/2$  for any barycenter  $p$ . Now, we construct a measurable open set  $U_n$  that contains  $\mathcal{T}'^n$  and such that each of its connected components contains only one  $n$ -face and is contained in the respective geodesic ball. This measurable open set contracts to the set of barycenters of  $\mathcal{T}'^n$  by the exponential map, which completes the proof.

Let  $\mathcal{B}^n$  denote the set of barycenters of  $\mathcal{T}'^n$ . Let  $\rho : e(p) \rightarrow \mathbb{R}^+$  be a continuous function and let  $N(e(p), \rho)$  denote the neighborhood of  $e(p)$  consisting of the union

of the balls of radius  $\rho(x)$  in the geodesic orthogonal sections of  $e(p)$  through  $x$ . We define  $h^n : \mathcal{T}^n \rightarrow \mathbb{R}$  by  $h(x) = d_g(x, \mathcal{T}'^n \setminus e(x))$ , which is measurable since  $g$  and  $\mathcal{T}'$  are measurable. Now, let  $\rho_p^n : e(p) \rightarrow \mathbb{R}^+$  be given by

$$\rho_p^n(x) = \frac{1}{2} \min\{h(x), i(p)\}.$$

Clearly,  $U^n = \bigcup_{p \in \mathcal{B}^n} N(e(p), \rho_p^n)$  is a measurable open set that covers  $\mathcal{T}'^n$ . Each open set  $N(e(p), \rho_p^n)$  is contained in the maximal exponential ball centered at  $p$  by definition of  $\rho_p^n$  and the conditions satisfied by  $\mathcal{T}'$ . These open sets are disjoint from each other. In fact, if  $x \in N(e(p), \rho_p^n) \cap N(e(p'), \rho_{p'}^n)$ , then

$$d_g(x, e(p)) = d_g(x, \xi) \leq \frac{1}{2} d_g(x, \mathcal{T}'^n \setminus e(p)) \leq \frac{1}{2} d_g(x, e(p')) = \frac{1}{2} d_g(x, \xi')$$

for certain  $\xi \in e(p)$  and  $\xi' \in e(p')$ . By the symmetric argument,  $d_g(x, \xi') \leq \frac{1}{2} d_g(x, \xi)$ , which is contradiction. Therefore the exponential map defines a measurable contraction of the measurable open set  $U^n$  to  $\mathcal{B}'^n$ .  $\square$

**Theorem 7.12** (Dimensional upper bound). *Let  $T$  be a complete transversal for the  $C^1$  measurable lamination  $(X, \mathcal{F}, \Lambda)$  with a  $C^1$  measurable triangulation. Then  $\text{Cat}(\mathcal{F}, \Lambda) \leq (\dim \mathcal{F} + 1) \cdot \Lambda(T)$ .*

*Proof.* Measurable laminations of class  $C^1$  admit a  $C^1$  measurable triangulation and a leafwise Riemannian metric [3]. By the Proposition 7.11, there exists a categorical measurable open set  $U^n$  containing each set  $\mathcal{T}'^n$  associated to a measurable triangulation for  $0 \leq n \leq \dim \mathcal{F}$ . Hence the sets  $U^n$  cover  $X$ . By Proposition 7.10,  $\tau_\Lambda(U^n) \leq \Lambda(T)$  for  $0 \leq n \leq \dim \mathcal{F}$ .  $\square$

This theorem has important consequences.

**Corollary 7.13.** *Let  $(X, \mathcal{F})$  a minimal  $C^1$  lamination. Let  $\Lambda$  be a regular transverse invariant measure of  $\mathcal{F}$  without atoms. Then  $\text{Cat}(\mathcal{O}(\mathcal{F}), \Lambda) = 0$ .*

Recall that a transverse invariant measure of a foliated measurable space is called *ergodic* if it is finite in a complete transversal and any saturated measurable set has null or full measure.

**Corollary 7.14.** *Let  $(X, \mathcal{F}, \Lambda)$  be a  $C^1$  measurable lamination with an ergodic transverse invariant measure without atoms. Then  $\text{Cat}(\mathcal{F}, \Lambda) = 0$ .*

*Proof.* Of course, in a ergodic lamination without atoms there exists complete transversals with arbitrarily small measure.  $\square$

## 8. COHOMOLOGICAL LOWER BOUND

In this section, we give a version of the useful cohomological lower bound of the classical LS category, stating that  $\text{Nil}(H^*(M, \Gamma)) \leq \text{Cat}(M)$  for any manifold  $M$ , where  $\text{Nil}(H^*(M, \Gamma))$  denotes the nilpotence order of the cohomology ring  $H^*(M, \Gamma)$  with coefficients in any ring  $\Gamma$  [9]. For this purpose, we give an idea of the cohomology of MT-spaces [2, 13, 18].

We suppose that  $\Gamma$  is a standard ring; i.e.,  $\Gamma$  is a standard space and a ring where all the operations are measurable.

**Definition 8.1** (Measurable prism [2, 3]). A *measurable prism* is a product of a standard Borel space  $T$  and a linear region of  $\mathbb{R}^N$  (for instance a polygon) with the standard MT-structure. A *measurable simplex* is a measurable prism where the topological fiber is a canonical  $n$ -simplex  $\Delta^n$ . A *measurable singular simplex* on  $X$  is an MT-map  $\sigma : \Delta^n \times T \rightarrow X$ .

Let  $\omega$  be a usual *singular  $n$ -cochain* over a coefficient ring  $\Gamma$ . It is said that  $\omega$  is *measurable* if  $\omega_\sigma : T \rightarrow \Gamma$ ,  $t \mapsto \omega(\sigma|_{\Delta \times \{t\}})$ , is measurable for all measurable singular  $n$ -simplex  $\sigma$ . The set of measurable cochains form a subcomplex of the complex of usual cochains since the coboundary operator  $\delta$  preserves measurability. This is called the *measurable subcomplex* and denoted by  $C_{MT}^*(X, \Gamma)$ , and the corresponding restriction of  $\delta$  is denoted by  $\tilde{\delta}$ . The *measurable singular cohomology* is defined as usual by  $H_{MT}^n(X, \Gamma) = \text{Ker } \tilde{\delta}_n / \text{Im } \tilde{\delta}_{n-1}$ .

The usual cup product induces a product on measurable cochains, which of course preserves measurability since the operations in  $\Gamma$  are measurable. The usual formula  $\tilde{\delta}(\omega \smile \theta) = \tilde{\delta}\omega \smile \theta + (-1)^n \omega \smile \tilde{\delta}\theta$  holds. Therefore it induces a cup product in measurable cohomology, obtaining the graded ring  $(\bigoplus_{n \geq 1} H_{MT}^n(X, \Gamma), +, \smile)$ .

Let  $f : X \rightarrow Y$  be an MT-map. Clearly,  $f$  induces a cochain map  $f^* : C_{MT}^*(Y, \Gamma) \rightarrow C_{MT}^*(X, \Gamma)$  defined by  $f^*(\omega)(\sigma) = \omega(f \circ \sigma)$ . This cochain map commutes with  $\tilde{\delta}$ , and therefore it induces a homomorphism between the corresponding measurable cohomology groups.

Let  $U \subset X$  be an MT-subspace of  $X$ . The inclusion map determines a chain map  $i^* : C_{MT}^*(X, \Gamma) \rightarrow C_{MT}^*(U, \Gamma)$ . The cochain complex determined by  $\text{Ker}(i^*)$  will be denoted by  $C_{MT}^*(X, U, \Gamma)$ . The cochains in this cochain complex are usual cochains vanishing on the singular simplices contained in  $U$ . The cohomology groups associated to this chain complex will be called the *measurable relative cohomology* groups of  $(X, U)$ . By using the Ker-CoKer Lemma, there exists a long exact sequence of cohomology groups like in the classical case (the details are easy to check),

$$\cdots \rightarrow H_{MT}^n(X, U, \Gamma) \rightarrow H_{MT}^n(X, \Gamma) \rightarrow H_{MT}^n(U, \Gamma) \rightarrow H_{MT}^{n+1}(X, U, \Gamma) \rightarrow \cdots$$

**Proposition 8.2** (Invariance by measurable tangential homotopy). *Let  $f, g : X \rightarrow Y$  be MT-homotopic maps. Then  $f^*$  and  $g^*$  induce the same homomorphism in measurable singular cohomology.*

*Proof.* The proof is a trivial consequence of the classical proof for singular cohomology. The measurable homotopy induces a chain homotopy between  $f_*$  and  $g_*$  at the level of the chain complex. The definition is given by cutting the space  $\Delta^n \times [0, 1]$  into a finite number of  $n+1$ -prisms  $\Pi_i : \Delta^{n+1} \rightarrow \Delta^n \times [0, 1]$ . Let  $H$  be the measurable homotopy between  $f$  and  $g$ . The prism operator  $P : C^{n+1}(X, \Gamma) \rightarrow C^n(X, \Gamma)$ , defined by  $P(\omega)(\sigma) = \sum_i I_i \omega(H \circ \sigma|_{\Pi_i})$  [12], where  $I_i$  is the orientation factor, is a cochain homotopy between  $f^*$  and  $g^*$  preserving measurability. Hence  $f^*$  and  $g^*$  induce the same homomorphism in measurable cohomology.  $\square$

Let  $C_{MT}^*(U + V)$  be the cochain complex given by the measurable cochains which vanish on measurable singular simplices that do not lie in either  $U$  or  $V$ . Now,  $H_{MT}^*(\mathcal{F}_{U \cup V})$  is isomorphic to  $H_{MT}^*(U + V)$  via the restriction map [18] like in the usual singular cohomology. In fact, by using the 5-Lemma,  $H_{MT}^*(\mathcal{F}, \mathcal{F}_U)$  is isomorphic to  $H_{MT}^*(\mathcal{F}, U + V)$ .

An easy computation shows that  $H_{\text{MT}}^n(X, \Gamma) = 0$  for  $n \geq 1$  when  $X$  is an MT-space with the discrete topology. That is the case of any measurable transversal of a measurable lamination.

**Corollary 8.3.** *The cup product induces a cup product in measurable relative cohomology:*

$$\smile : H_{\text{MT}}^n(\mathcal{F}, \mathcal{F}_U, \Gamma) \times H_{\text{MT}}^m(\mathcal{F}, \mathcal{F}_V, \Gamma) \rightarrow H_{\text{MT}}^{n+m}(\mathcal{F}, \mathcal{F}_{U \cup V}, \Gamma).$$

**Corollary 8.4.**  $\text{Nil}(\bigoplus_{n \geq 1} H_{\text{MT}}^n(\mathcal{F}, \Gamma), +, \smile) \leq \text{Cat } \mathcal{F}.$

*Proof.* Let  $\{U_1, \dots, U_N\}$  be a covering by tangentially contractible measurable open sets. The map  $H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}_{U_i}, \Gamma) \rightarrow H_{\text{MT}}^*(\mathcal{F}, \Gamma)$  of the cohomological exact sequence is onto since each  $U_i$  is categorical. Let  $x_1, \dots, x_N$  be cohomology classes in  $H_{\text{MT}}^*(\mathcal{F})$ , and take a preimage  $\omega_i$  of each  $x_i$  in  $H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}_{U_i}, \Gamma)$ . Therefore  $\omega_1 \smile \dots \smile \omega_N \in H_{\text{MT}}^*(\mathcal{F}, \mathcal{F}, \Gamma) = 0$ , and this product projects to  $x_1 \smile \dots \smile x_N$  in  $H_{\text{MT}}^*(\mathcal{F}, \Gamma)$ .  $\square$

Of course, singular measurable cohomology is a complicated object. Fortunately, we can work with simplicial measurable cohomology [2, 13, 18].

Let  $\mathcal{T}$  be a measurable triangulation. An  $n$ -cochain over a measurable ring  $\Gamma$  is a measurable map  $\omega : \mathcal{B}^n \rightarrow \Gamma$ , where the barycenters in  $\mathcal{B}^n$  are identified to the corresponding  $n$ -simplices. Let  $C^n(\mathcal{T}, \Gamma)$  denote the set of simplicial  $n$ -cochains, which is endowed with a ring structure induced by  $\Gamma$ . Define the coboundary operator  $\delta : C^n(\mathcal{T}, \Gamma) \rightarrow C^{n+1}(\mathcal{T}, \Gamma)$  as usual: for  $\omega : \mathcal{B}_{n+1} \rightarrow \Gamma$ , let  $\delta\omega : \mathcal{B}_{n+1} \rightarrow \Gamma$  be given by  $\delta\omega(b) = \sum_{\Delta_p^n \subset \partial \Delta_b^{n+1}} (-1)^n \omega(p)$ , where  $\Delta_x^k$  denotes the  $k$ -simplex with barycenter  $x$  and the simplices of the boundary are chosen in the usual order [12]. Clearly,  $\delta^2 = 0$ , and we can define the cohomology groups as usual:  $H^n(\mathcal{T}, \Gamma) = \text{Ker } d_n / \text{Im } d_{n-1}$ .

**Proposition 8.5** ([18]). *Let  $(X, \mathcal{F})$  be a measurable lamination that admits a measurable triangulation. Then the measurable singular cohomology groups are isomorphic to the measurable simplicial ones.*

**Corollary 8.6.** *The measurable simplicial cohomology does not depend on the measurable triangulation.*

*Remark 7.* Of course, we can define the concept of measurable CW-structure in a similar way to a measurable triangulation. This notion gives a measurable CW-complex and a measurable cellular cohomology. It can be proved that it is isomorphic to the measurable singular cohomology with arguments similar to the above ones.

**Example 8.7.** Let  $(T^2, \mathcal{F}_\alpha)$  be the K onecker flow, considered as a suspension of the rotation  $R_\alpha : S^1 \rightarrow S^1$  of  $2\pi\alpha$  radians. The case where  $\alpha$  is rational is trivial (it is a foliation with compact leaves). Then suppose that  $\alpha$  is irrational. The projection of  $[0, 1] \times S^1$  to  $T^2$ , given by the suspension of  $R_\alpha$ , induces a measurable triangulation of  $\mathcal{F}_\alpha$ , where the 0-skeleton is the projection of  $\{0\} \times S^1$  and the 1-skeleton is the projection of  $[0, 1] \times S^1$ ; the set of barycenters is the projection of  $\{1/2\} \times S^1$ . Of course, measurable cochains of degrees 0 and 1 are measurable maps  $f : S^1 \rightarrow \mathbb{Z}_2$ . In [18], it is proved that the 1-cochain  $\omega = 1 : S^1 \rightarrow \mathbb{Z}_2$  represents a non-zero element in  $H_{\text{MT}}^1(\mathcal{F}_\alpha, \mathbb{Z}_2)$ , showing that  $H^1(\mathcal{F}_\alpha, \mathbb{Z}_2) \neq 0$ . By Corollary 8.4 and the dimensional bound, it follows that  $\text{Cat}(\mathcal{F}_\alpha) = 2$ .



In higher dimension, let  $\mathcal{F}_{\alpha_1, \dots, \alpha_n}$  be the foliation on  $T^{n+1}$  given by the suspension of the rotations  $R_{\alpha_1}, \dots, R_{\alpha_n}$  of  $S^1$ , where  $\alpha_1, \dots, \alpha_n$  are  $\mathbb{Q}$ -linear independent. Each leaf of  $\mathcal{F}_{\alpha_1, \dots, \alpha_n}$  is a hyperplane dense in  $T^{n+1}$ . Let  $[0, 1]^n \subset \mathbb{R}^n$  be the unit cube and let  $T \equiv S^1$ . The product  $[0, 1]^n \times T$  gives a *measurable CW-structure* on  $\mathcal{F}$  given by the projection  $p : [0, 1]^n \times T \rightarrow \mathcal{F}$  defined by the suspension. Let  $\omega = 1 : T \rightarrow \mathbb{Z}_2$ , which is a CW-cochain of dimension  $n$ . Let  $\tau_i$  the CW-cochain of dimension 1 satisfying  $\tau_i(p|_{[0, 1]_j \times T}) = \delta_{ij}$ , where

$$[0, 1]_j = \{ (x_1, \dots, x_n) \in [0, 1]^n \mid x_i = 0 \text{ for } i \neq j \}.$$

Clearly  $\omega = \tau_1 \smile \dots \smile \tau_n$ . Moreover  $\omega$  represents a non-zero element in  $H_{\text{MT}}^n(\mathcal{F}_{\alpha_1, \dots, \alpha_n}, \mathbb{Z}_2)$  [18], obtaining  $H_{\text{MT}}^n(\mathcal{F}_{\alpha_1, \dots, \alpha_n}, \mathbb{Z}_2) \neq 0$ . Then Corollary 8.4 and the dimensional upper bound yields  $\text{Cat}(\mathcal{F}_{\alpha_1, \dots, \alpha_n}) = n + 1$ .

*Remark 8.* We do not discuss here about the possibility of a similar lower bound for the  $\Lambda$ -category. In many general situations, this invariant is zero and a lower bound is not so interesting. In the cases where there exists a complete transversal  $T$  meeting each leaf at exactly one point, the number  $\int_T \text{Nil } H^*(L_t, \Gamma) d\Lambda(t)$  is well defined and gives a lower bound for the  $\Lambda$ -category.

## 9. CRITICAL POINTS

As suggested by the classical theory, because of the possible applications, it is interesting to consider Hilbert laminations to study the relation between our measurable versions of the tangential category and the critical points of a smooth function.

Recall the following terminology. A *Hilbert manifold* is a separable, Hausdorff space endowed with an atlas where the charts are homeomorphisms to open subsets of separable Hilbert spaces. A  $C^2$  function on a complete  $C^2$  Hilbert manifold,  $f : M \rightarrow \mathbb{R}$ , is called *Palais-Smale* whenever, for any sequence  $(p_n)$  in  $M$ , if  $(f(p_n))$  is bounded and  $(df(p_n))$  converges to zero, then  $(p_n)$  contains a convergent subsequence. If  $M$  is compact (in the finite dimensional case), then all differentiable maps are Palais-Smale. Let  $\text{Crit}(f)$  be the set of critical points of  $f$ . In this section, we adapt a theorem due to J. Schwartz [21], which states that, for a bounded from below Palais-Smale function on a  $C^2$  Hilbert manifold,  $f : M \rightarrow \mathbb{R}$ , we have  $\text{Cat}(M) \leq \# \text{Crit}(f)$ .

For the rest of the section, we consider  $C^2$  measurable Hilbert laminations. Their definition is analogous to the definition of measurable lamination: the leafwise topological model is now given by open balls in a separable Hilbert space instead of  $\mathbb{R}^n$ , and the tangential part of the changes of coordinates are  $C^2$  maps between open subsets of Hilbert spaces. Each leaf is a Hilbert manifold, which is endowed with a Riemannian metric that varies in a measurable way in the ambient space. Observe that the main difference with the finite dimensional case is that the leaves may not be locally compact. We also suppose that the leafwise Riemannian metric is complete.

To define the  $\Lambda$ -category of a measurable Hilbert lamination with a transverse invariant measure  $\Lambda$ , we consider only contractible measurable open sets. The reason is that Proposition 4.4 seems to be difficult to generalize to this infinite dimensional setting, as well as other details about measurability. However, Proposition 4.3 holds for measurable Hilbert laminations too, and therefore we can use the coherent extension  $\tilde{\Lambda}$  of  $\Lambda$ .

Notice that the differential map of a function varies in a measurable way in the ambient space, since its definition is a limit of measurable maps. The set of  $C^2$  MT-functions  $\mathcal{F} \rightarrow \mathbb{R}$  will be noted by  $C^2(\mathcal{F})$ . For  $f \in C^2(\mathcal{F})$ , let  $\text{Crit}_{\mathcal{F}}(f) = \bigcup_{L \in \mathcal{F}} \text{Crit}(f|_L)$ .

The cotangent bundle  $T\mathcal{F}^*$  is a measurable vector bundle, whose zero section  $\theta : X \rightarrow T\mathcal{F}^*$  is measurable with measurable image. For any  $f \in C^2(\mathcal{F})$ , its differential map  $df : X \rightarrow T\mathcal{F}^*$  is measurable, and we have  $\text{Crit}_{\mathcal{F}}(f) = df^{-1}(\theta(X))$ . Thus  $\text{Crit}_{\mathcal{F}}(f)$  is measurable.

Recall that a  $C^1$  isotopy on a  $C^1$ -Hilbert manifold  $M$  is a differentiable map  $\phi : M \times \mathbb{R} \rightarrow M$  such that  $\phi_t = \phi(\cdot, t) : M \rightarrow M$  is a diffeomorphism  $\forall t \in [0, 1]$  and  $\phi_0 = \text{id}_M$ .

**Definition 9.1** (Measurable tangential isotopy). Let  $(X, \mathcal{F})$  be a  $C^1$  measurable Hilbert lamination. A *measurable tangential isotopy* on  $(X, \mathcal{F})$  is a  $C^1$  map  $\phi : X \times \mathbb{R} \rightarrow X$  such that the functions  $\phi_t : X \rightarrow X$  are MT-diffeomorphisms  $\forall t$ , with  $\phi_0 = \text{id}_X$ , and the map  $\phi : \mathcal{F} \times \mathbb{R} \rightarrow \mathcal{F}$ , with  $\phi(x, t) = \phi_t(x)$ , is  $C^1$ , where  $\mathcal{F} \times \mathbb{R}$  is the  $C^1$  measurable lamination in  $X \times \mathbb{R}$  with leaves of the form  $L \times \mathbb{R}$  for  $L \in \mathcal{F}$ . In particular, the maps  $\phi^x : \mathbb{R} \rightarrow X$ , with  $\phi^x(t) = \phi_t(x)$ , are differentiable  $\forall x \in X$ .

*Remark 9.* Let  $\phi$  be a measurable tangential isotopy on  $X$  and let  $U \subset X$  be a measurable open set. Then  $\text{Cat}(U, \mathcal{F}, \Lambda) = \text{Cat}(\phi_t(U), \mathcal{F}, \Lambda)$  and  $\text{Cat}(U, \mathcal{F}) = \text{Cat}(\phi_t(U), \mathcal{F})$  for all  $t \in \mathbb{R}$ .

**Example 9.2** (Construction of a measurable tangential isotopy [19]). A tangential isotopy can be constructed on a Hilbert manifold by using a  $C^1$  tangent vector field  $V$ . There exists a flow  $\phi_t(p)$  such that  $\phi_0(p) = p$ ,  $\phi_{t+s}(p) = \phi_t(\phi_s(p))$  and  $d\phi_t(p)/dt = V(\phi_t(p))$ . From the way of obtaining  $\phi$  [19, 8], it follows that the same kind of construction for a measurable  $C^1$  tangent vector field on a measurable Hilbert lamination  $(X, \mathcal{F})$  induces a measurable isotopy on  $(X, \mathcal{F})$ .

Now, we obtain a measurable isotopy from the gradient flow. It will be modified by a control function  $\alpha$  in order to have some control in the deformations induced by the corresponding isotopy. Let  $\nabla f$  be the gradient tangent vector field of  $f$ ; i.e., the unique tangent vector field satisfying  $df(v) = \langle v, \nabla f \rangle$  for all  $v \in T\mathcal{F}$ . Take the  $C^1$  vector field  $V = -\alpha(|\nabla f|) \nabla f$ , where  $\alpha : [0, \infty) \rightarrow \mathbb{R}^+$  is  $C^\infty$ ,  $\alpha(t) \equiv 1$  for  $0 \leq t \leq 1$ ,  $t^2\alpha(t)$  is monotone non-decreasing and  $t^2\alpha(t) = 2$  for  $t \geq 2$ . The flow  $\phi_t(p)$  of  $V$  is defined for  $-\infty < t < \infty$  [21], and it is called the *modified gradient flow*.

Let us define a partial order relation “ $\ll$ ” for the critical points of  $f$ . First, we say that  $x < y$  if there exists a regular point  $p$  such that  $x \in \alpha(p)$  and  $y \in \omega(p)$ , where  $\alpha(p)$  and  $\omega(p)$  are the  $\alpha$ - and  $\omega$ -limits of  $p$ . Then  $x \ll y$  if there exists a finite sequence of critical points,  $x_1, \dots, x_n$ , such that  $x < x_1 < \dots < x_n < y$ .

Let  $\gamma(x)$  denote the  $\phi$ -orbit of each point  $x$ .

**Lemma 9.3.** *Let  $T \subset X$  be an isolated transversal. Then there exists a measurable open set  $U$  containing  $T$  such that  $\text{Cat}(U, \mathcal{F}, \Lambda) \leq \Lambda(T)$ . We can suppose also that  $U$  is tangentially categorical contracting to  $T$ .*

*Proof.* A tubular neighborhood of  $T$  contracts to  $T$  (see Lemma 7.8 and observe that its proof generalizes to measurable Hilbert laminations). Hence its relative category is less or equal than  $\Lambda(T)$ .  $\square$

In the following proposition,  $\{W_1, W_2, \dots\}$  denotes a foliated measurable atlas. Let  $\text{Crit}_{\mathcal{F}}^{\infty}(f)$  be the union of plaques that contain infinite critical points of  $f$ ; observe that this is a measurable open set.

**Proposition 9.4.** *If  $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}^{\infty}(f)) > 0$ , then  $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f)) = \infty$ .*

*Proof.* For each chart  $W_i$ , let  $\pi_i : W_i \rightarrow T_i$  be the transverse projection. Since  $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}^{\infty}(f)) > 0$ , we have  $\Lambda(\pi_i(\text{Crit}_{\mathcal{F}}^{\infty}(f) \cap W_i)) > 0$  for some  $i \in \mathbb{N}$ . Therefore

$$\begin{aligned} \tilde{\Lambda}(\text{Crit}_{\mathcal{F}}^{\infty}(f)) &\geq \int_{\pi_i(\text{Crit}_{\mathcal{F}}^{\infty}(f) \cap W_i)} \#(\text{Crit}_{\mathcal{F}}(f) \cap \pi_i^{-1}(t)) d\Lambda(t) \\ &= \infty \cdot \Lambda(\pi_i(\text{Crit}_{\mathcal{F}}^{\infty}(f) \cap W_i)) = \infty. \quad \square \end{aligned}$$

*Remark 10.* The set  $\text{Crit}_{\mathcal{F}}^{\infty}(f)$  contains all non-isolated critical points of  $\text{Crit}_{\mathcal{F}}(f)$ . If  $\tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f)) < \infty$ , then the saturation of  $\text{Crit}_{\mathcal{F}}^{\infty}(f)$  is a null-transverse set. Hence we can restrict our study to the case where all critical points are isolated.

The definition of a Palais-Smale condition is needed for a version of the Lusternik-Schnirelmann Theorem on Hilbert manifolds. For measurable Hilbert laminations, it could be adapted by taking functions that satisfy the Palais-Smale condition on all (or almost all) leaves. But this is very restrictive because it would mean that the set of relative minima meets each leaf in a relatively compact set (which is non-empty when  $f$  is bounded from below), and therefore there would exist a complete transversal meeting each leaf at one point. Thus, instead, we use the following weaker condition.

**Definition 9.5.** A measurable  $\omega$ -Palais-Smale (or  $\omega$ -PS) function is a function  $f \in C^2(\mathcal{F})$  such that any  $\phi$ -orbit have non empty  $\omega$ -limit and, for any  $p \in \text{Crit}_{\mathcal{F}}(f)$ , the set  $\{x \in \text{Crit}_{\mathcal{F}}(f) \mid p \ll x\}$  is compact, and this set is empty if and only if  $p$  is a relative minimum. A measurable  $\alpha$ -Palais-Smale (or  $\alpha$ -PS) function is defined analogously by considering the set  $\{x \in \text{Crit}_{\mathcal{F}}(f) \mid x \ll p\}$ .

Of course,  $f$  is  $\omega$ -PS if and only if  $-f$  is  $\alpha$ -PS.

**Lemma 9.6.** *Suppose that  $\text{Crit}_{\mathcal{F}}(f)$  is an isolated transversal. The modified gradient flow  $\phi$  (see Example 9.2) satisfies the following properties:*

- (i) *The flow runs towards lower level sets of  $f$ , i.e.,  $f(p) \geq f(\phi_t(p))$  for  $t > 0$ .*
- (ii) *The invariant points of the flow are just the critical points of  $f$ .*
- (iii) *A point is critical if and only if  $f(\phi_t(p)) = f(p)$  for some  $t \neq 0$ .*
- (iv) *The points in the  $\alpha$ - and  $\omega$ -limits are critical points if they are non empty.*

*Proof.* These properties can be proved in each leaf, considered as a  $C^2$  Hilbert manifold, where (i), (ii) and (iii) follow from the work of J. Schwartz [21].

Under these conditions, the  $\alpha$ - and  $\omega$ -limits are connected sets that consist of critical points if they are non-empty (by using (i), (ii) and (iii)). If  $\omega(p)$  is infinite, then all of its points are non-isolated, contradicting the assumption.  $\square$

**Definition 9.7** (Critical sets). For a measurable  $\omega$ -PS function, the set of minima is non-empty in any leaf. Let  $p$  be a critical point. By the properties of the flow  $\phi$ , either  $p$  is a relative minimum, or there exists another critical point  $x$  such that  $p < x$ . Define  $M_0, M_1, \dots$  inductively by

$$\begin{aligned} M_0 &= \{x \in \text{Crit}_{\mathcal{F}}(f) \mid \nexists y \text{ such that } x \ll y\}, \\ M_i &= \{x \in \text{Crit}_{\mathcal{F}}(f) \mid \forall y \ x \ll y \Rightarrow y \in M_0 \cup \dots \cup M_{i-1}\}. \end{aligned}$$

Clearly,  $M_0$  contains all relative minima on the leaves. We also set  $C_0(f) = M_0$  and  $C_i(f) = M_i \setminus (M_0 \cup \dots \cup M_{i-1})$ . Observe that, if  $x \in \omega(p)$  and  $x \in C_i(f)$  for some  $i$ , then  $\omega(p) \subset C_i(f)$ . There is an analogous property for the  $\alpha$ -limit. The notation  $C_i$  will be used if there is no confusion. Let  $p \ll p^*$ . Then  $i_{p^*} < i_p$ , where  $i_p$  and  $i_{p^*}$  are the indexes such that  $p \in C_{i_p}$  and  $p^* \in C_{i_{p^*}}$ .

The set of relative minima of a bounded from below measurable  $\omega$ -PS function is always non-empty in any leaf.

**Theorem 9.8.** *Let  $(X, \mathcal{F})$  be a measurable Hilbert lamination endowed with a measurable Riemannian metric on the leaves, and let  $f$  be a measurable  $\omega$ -PS function on  $X$ . Suppose that  $\text{Crit}_{\mathcal{F}}(f)$  is an isolated transversal. Then  $\text{Cat}(\mathcal{F}) \leq \#\{\text{critical sets of } f\}$ .*

**Theorem 9.9.** *Let  $(X, \mathcal{F}, \Lambda)$  be a measurable Hilbert lamination endowed with a measurable Riemannian metric on the leaves and with a transverse invariant measure, and let  $f$  be a measurable  $\omega$ -PS function on  $X$ . Then  $\text{Cat}(\mathcal{F}, \Lambda) \leq \tilde{\Lambda}(\text{Crit}_{\mathcal{F}}(f))$ .*

*Remark 11.* Notice also that Theorem 9.8 gives a slight sharpening of the classical theorem of Lusternik-Schnirelmann, since the number of critical sets may be finite even when the number of all critical points are infinite. We see this in the following example. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \sin(x)$ , which is not an Palais-Smale function in the classical sense, but it satisfies our measurable Palais-Smale condition in the one leaf lamination  $\mathbb{R}$ . An easy computation shows that there are two critical sets, the set of relative minima and the set of relative maxima, yielding the inequality  $\text{Cat}(\mathbb{R}) \leq 2$ . On more complicated examples, this improved version could be used to find better upper bounds of the classical LS category.

*Remark 12.* From the existence of a measurable Riemannian metric, two disjoint isolated transversals, can be separated by measurable open sets. In fact, we can suppose that the closures of the connected components of these measurable open sets contain only one point of these transversals (see Proposition 7.3).

**Lemma 9.10.** *Let  $(X, \mathcal{F})$  be a measurable Hilbert lamination and let  $f_n : (X, \mathcal{F}) \rightarrow (X, \mathcal{F})$  be a sequence of MT-maps. Suppose that  $(f_n(x))$  converges for all  $x \in X$ . Then  $\lim_n f_n$  is measurable.*

*Proof.* Measurable open sets generate the  $\sigma$ -algebra of  $(X, \mathcal{F})$ , in fact, by Theorem ?? the measurable foliated charts are a generating set also. Then it is enough to prove that  $(\lim_n f_n)^{-1}(V)$  is measurable for any foliated chart  $V$ . For each  $V \equiv B \times T$  there exists a sequence of measurable closed sets  $\{F_n\}_{n \in \mathbb{N}}$  such that  $V = \bigcup_n F_n$ . For instance, if  $B$  is an open ball we can take  $F_n \equiv \overline{B_n} \times T$ , where  $B_n$  are open balls of smaller radius than  $B$  but converging to it. Now, it is clear that

$$\begin{aligned} (\lim_n f_n)^{-1}(V) &= \{x \in X \mid \exists N \text{ such that } f_n(x) \in F_N \ \forall n \geq N\} \\ &= \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} f_n^{-1}(F_N), \end{aligned}$$

which is, clearly, a measurable set.  $\square$

*Remark 13.* It is known, in basic measure theory, that the limit of real measurable functions is also measurable. Lemma 9.10 is not a direct consequence from this fact

because the measurable structure is not the Borel  $\sigma$ -algebra corresponding to the topology.

**Corollary 9.11.** *The family  $\{C_i\}$  is a measurable partition of  $\text{Crit}_{\mathcal{F}}(f)$ .*

*Proof.* Clearly, this family is a partition by the properties of  $\phi$ . To show that each  $C_i$  is measurable, observe that  $\text{Crit}_{\mathcal{F}}(f)$  is a transversal, and  $X \setminus \text{Crit}_{\mathcal{F}}(f)$  is measurable with open intersection with each leaf. The  $\alpha$ - and  $\omega$ -limit functions,  $\alpha, \omega : X \rightarrow \text{Crit}_{\mathcal{F}}(f)$ , are measurable by Lemma 9.10 since  $\alpha = \lim_n \phi_{-n}$  and  $\omega = \lim_n \phi_n$ . Observe that  $\alpha(X \setminus \text{Crit}_{\mathcal{F}}(f)) = \text{Crit}_{\mathcal{F}}(f) \setminus C_0$ . Therefore  $C_0$  is measurable. By Remark 12, there exists a measurable open set  $U_0$  containing  $C_0$  and separating it from  $\text{Crit}_{\mathcal{F}}(f) \setminus C_0$ ; in fact, each connected component contains only one point of  $C_0$ . Take the measurable open set  $O_0 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(U_0)$ . It is easy to see that

$$\alpha(X \setminus (O_0 \cup \text{Crit}_{\mathcal{F}}(f))) = \text{Crit}_{\mathcal{F}}(f) \setminus (C_0 \cup C_1).$$

Therefore  $C_1$  is a measurable set. By a recursive argument, we obtain that all sets  $C_i$  are measurable.  $\square$

**Lemma 9.12.** *Let  $(X, \mathcal{F})$  be a measurable Hilbert lamination and let  $(Y, \mathcal{G})$  be a finite dimensional lamination such that  $Y \subset X$  is a measurable open set and the inclusion map is an MT-embedding considering in  $Y$ . Then there exists a countable family of measurable foliated charts of  $(X, \mathcal{F})$ ,  $\{U_n \equiv B_n \times T_n\}_{n \in \mathbb{N}}$ , covering  $Y$  and such that each fiber  $B_n \times \{t\}$  is a foliated chart (in a topological sense) of  $\mathcal{G}$  in a leaf of  $\mathcal{F}_Y$ .*

*Proof.* Let  $U$  be a measurable foliated chart of  $\mathcal{F}$ . Since  $Y$  is open in  $X$ ,  $U \cap Y$  is also measurable and open. Hence a measurable foliated atlas induces a covering of  $Y$  by measurable open sets and a measurable foliated atlas of  $(Y, \mathcal{F}_Y)$ . Since  $\mathcal{G}$  is a sublamination of  $\mathcal{F}_Y$ , by Theorem 2.2, we can choose a measurable foliated atlas such that the plaques are products of the form  $B^k \times B'$  where  $B^k$  are open balls in  $\mathbb{R}^k$  where  $\dim \mathcal{G} = k$  and  $B'$  is a ball in the orthogonal complement of  $\mathcal{G}$  in a leaf of  $\mathcal{F}$  centered in the origin. Of course the projection  $\pi : (B^k \times B) \times T \rightarrow (B^k \times \{0\}) \times T$  defines an MT-map that works like a tubular neighborhood of a plaque of a family of leaves of  $\mathcal{G}$ . Of course  $B^k \times (B' \times T)$  is a measurable chart for  $\mathcal{G}$ , hence the family of tubular neighborhoods is a measurable atlas of  $\mathcal{G}$  and  $\mathcal{F}_Y$  simultaneously.  $\square$

*Proof of Theorem 9.8.* By Remark 12 and Lemma 9.3, there exists a disjoint family of measurable open sets,  $\{U_i\}$  ( $i \in \mathbb{N} \cup \{0\}$ ), such that each  $U_i$  contains and contracts to  $C_i$ , and  $\text{Cat}(U_i, \mathcal{F}, \Lambda) \leq \Lambda(C_i)$ . We also assume that  $\overline{U_i} \subset \widetilde{U_i}$ , where  $\widetilde{U_i}$  is another categorical measurable open set that contracts to  $C_i$ .

Let  $U'_0 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(U_0)$ . This set is open since  $C_0(f)$  consists of relative minima, and contracts to  $C_0(f)$  by using the MT isotopy  $\phi$ . The set  $X_1 = X \setminus U'_0$  is a measurable closed set, and  $C_1(f)$  is the set of relative minima of the restriction  $f|_{X_1}$ . The set  $X_1$  consists of the critical points that do not belong to  $C_0(f)$  and the regular points connecting these critical points according to the relation “ $\ll$ ”. Let  $F_1 = U_1 \cap X_1$  and  $F'_1 = \bigcup_{n \in \mathbb{N}} \phi_{-n}(F_1)$ . The set  $F'_1$  is open in  $X_1$  and closed in  $X$ . Let us prove that  $F'_1$  is contained in a measurable open set  $U'_1$  such that there exists a measurable deformation  $H$  with  $H(U'_1 \times \{1\}) \subset \widetilde{U}_1$ .

Of course,  $(X \setminus \text{Crit}_{\mathcal{F}}(f), \phi)$  is a measurable lamination of dimension 1, where the leaves are the flow lines of  $\phi$ . These flow lines are embedded submanifolds and they admit a countable covering by measurable foliated charts in the sense

of Lemma 9.12. Therefore there exists a measurable atlas of  $(X \setminus \text{Crit}_{\mathcal{F}}(f), \phi)$ ,  $\{(W_n, \varphi_n)\}_{n \in \mathbb{N}}$ , with  $\varphi_n : W_n \rightarrow B^1 \times B_n \times T_n$ , where  $B^1$  is an open interval in  $\mathbb{R}$ ,  $B_n$  is an open ball centered at the origin in a separable Hilbert space, and  $T_n$  is a standard space. We can suppose also that this measurable atlas is locally finite and let  $\pi_n : W_n \rightarrow \varphi_n^{-1}(B^1 \times \{0\} \times T_n)$  be given by the canonical projection  $B^1 \times B_n \times T_n \rightarrow B^1 \times \{0\} \times T_n$ .

By similar arguments, we can suppose that  $F'_1 \subset \bigcup_n W_n \subset \bigcup_{i \in \mathbb{N}} \phi_{-i}(U_1)$ ,  $\varphi_n(B^1 \times \{0\} \times T_n) \subset F'_1$  and  $\bigcup_n W_n$  is a *semisaturated* set; i.e., if  $x \in \bigcup_n W_n$  then  $\phi_t(x) \in \bigcup_n W_n$  for all  $t \in [0, \infty)$ . Let  $\{\lambda_n\}$  be a measurable partition of unity subordinated to  $\{W_n\}$  [3] such that each  $\lambda_n$  is continuous on  $\varphi_n^{-1}(B^1 \times B_n \times \{z\})$  for all  $z \in T_n$ . For each  $x \in \bigcup_n W_n$ , let  $I(x) \subset \mathbb{N}$  be the set of numbers  $n$  such that the semiorbit  $\phi_{[0, \infty)}(x)$  meets  $W_n$ . The isotopy  $\phi_t|_{F'_1}$  contracts  $F'_1$  to  $C_1$ . We extend the deformation  $\phi_t|_{F'_1 \setminus C_1(f)}$  to the neighborhood  $\bigcup_n W_n$ . This extension can be defined as follows: for  $x \in \bigcup_n W_n$ ,  $t \in \mathbb{R}$  and  $n \in I(x)$ , there is a unique positive real number  $r(x, t, n)$  such that  $\phi_{r(x, t, n)}(x) = \gamma(x) \cap \pi_n^{-1}(\phi_t(\pi_n(x)))$ . Let  $H : V_1 \times \mathbb{R} \rightarrow X$  be the continuous map defined by  $H(x, t) = \phi_{s(x, t)}(x)$ , where

$$s(x, t) = \sum_{k \in I(x)} \lambda_k(x) r(x, t, k) .$$

For  $x \in \bigcup_n W_n$  and  $t \in \mathbb{R}$ , there exists  $k_1, k_0 \in I(x)$  such that  $r(x, t, k_1) \leq s(x, t) \leq r(x, t, k_0)$ . It is clear that there exists  $\lim_{t \rightarrow \infty} \phi_{r(x, t, n)}(x) \in \overline{U_1} \subset \widetilde{U_1}$ . Let  $p \in C_1$  and let  $x \in F'_1 \setminus C_1$  with  $\omega(x) = p$ . Since  $\bigcup_n W_n$  is semisaturated and it is contained in  $\bigcup_{i \in \mathbb{N}} \phi_{-i}(U_1)$ , for all  $r(x, t, k_1) < t' < r(x, t, k_0)$ ,  $\phi_{t'}(x) \in \widetilde{U_1}$  for  $t$  large enough. Therefore  $\lim_{t \rightarrow \infty} H(x, t) \in \widetilde{U_1}$  for all  $x \in \bigcup_n V_n$ . Then the measurable open subset  $V'_1 = \bigcup_n V_n$  is  $\mathcal{F}$ -categorical (by a standard change of parameter). Finally, if  $\widetilde{U_1}$  is small enough,  $U'_1 = V'_1 \cup \widetilde{U_1}$  is  $\mathcal{F}$ -categorical by a telescopic argument [12] and  $F'_1 \subset U'_1$ .

This process can be continued inductively by taking  $X_n = X \setminus (U'_0 \cup \bigcup_{i=1}^{n-1} F'_i)$  and using the same trick to define  $U'_n$ , observing that  $C_n(f)$  is the set of relative minima of  $f|_{X_n}$ .  $\square$

*Proof of Theorem 9.9.* By Remark 10, we can restrict the study to the case where  $\text{Crit}_{\mathcal{F}}(f)$  is an isolated transversal. The previous proof also shows that  $\text{Cat}(\mathcal{F}, \Lambda)$  is a lower bound for the sum of the measures of the critical sets. Since the critical sets form a partition of  $\text{Crit}_{\mathcal{F}}(f)$ , the proof is complete.  $\square$

*Acknowledgment.* This paper contains part of my PhD thesis, whose advisor is Prof. Jesús A. Álvarez López.

## REFERENCES

- [1] W. BALLMAN. *Closed geodesics on positively curved manifolds*. Ann. of Math. 116, 213–247 (1982).
- [2] M. BERMÚDEZ. *Laminations Boréliennes*. Tesis, Université Claude Bernard-Lyon 1 (2004).
- [3] M. BERMÚDEZ, G. HECTOR. *Laminations hyperfinies et revêtements*. Ergod. Th. Dynam. Sys. 26, 305–339 (2006).
- [4] A. CANDEL, L. CONLON. *Foliations I*. Amer. Math. Soc. (1999)
- [5] H. COLMAN. *Categoría L-S en foliaciones*. Tesis, Departamento de Xeometría e Topoloxía, Universidad de Santiago de Compostela (1998).
- [6] H. COLMAN, E. MACÍAS VIRGÓS. *Tangential Lusternik-Schnirelmann category of foliations*. J. London Math. Soc. 2, 745–756 (2002).

- [7] A. CONNES. *A survey of foliations and operator algebras*. Proc. Sympos. Pure Math. 38, 520–628 (1982).
- [8] M.G. CRANDALL, A. PAZY. *Semi-groups of nonlinear contractions and dissipative sets*. J. Funct. Analysis 3, 376–418, (1969).
- [9] B. DUBROVIN, S. NOVIKOV, A. FOMENKO. *Modern Geometry -Methods and Applications: Part III: Introduction to Homology Theory*. Graduate Texts in Mathematics, Springer-Verlag (1990).
- [10] D.B.A. EPSTEIN. *Foliations with all leaves compact*. Ann. Inst. Fourier 26, 265–282 (1976).
- [11] ———. *Periodic flows on 3-manifolds*. Ann. Math. 95, 68–82 (1972).
- [12] A. HATCHER. *Algebraic Topology*. Cambridge University Press (2002).
- [13] J.L. HEITSCH, C. LAZAROV. *Homotopy invariance of foliation Betti numbers*. Invent. Math. 104, 321–347 (1991).
- [14] I.M. JAMES. *On category, in the sense of Lusternik-Schnirelmann*. Topology 17, 331–348 (1978).
- [15] R. KALLMAN. *Certain quotient spaces are countably separated, III*. J. Funct. Analysis 22, 225–241 (1976).
- [16] A.S. KECHRIS. *Classical Descriptive Set Theory*. Graduate Texts in Mathematics, Springer-Verlag, New York (1994).
- [17] C. MENIÑO. *Transverse invariant measures extend to the ambient space*. Preprint, arXiv:1103.4696v1.
- [18] ———. *Cohomology of measurable laminations*. Preprint, arXiv:1105.5948v1.
- [19] R.S. PALAIS. *Lectures on Morse Theory*. Lecture Notes, Brandies and Harvard Universities (1962).
- [20] L. LUSTERNIK, L. SCHNIRELMANN. *Méthodes Topologiques dans les Problemes Variationnels*. Herman, Paris (1934).
- [21] J.T. SCHWARTZ. *Generalizing the Lusternik-Schnirelmann Theory of Critical Points*. Comm. Pure Appl. Math. 17, 307–315 (1964).
- [22] W. SINGHOF, E. VOGT. *Tangential category of foliations*. Topology 42, 603–627 (2003).
- [23] S.M. SRIVASTAVA. *A course on Borel sets*. Graduate Texts in Mathematics, Springer, 1998.
- [24] M. TAKESAKI. *Theory of Operator Algebras*. Springer-Verlag, New York, Heidelberg, Berlin (1979).

DEPARTAMENTO DE GEOMETRÍA E TOPOLOGÍA, FACULTADE DE MATEMÁTICAS, UNIVERSIDADE DE SANTIAGO DE COMPOSTELA, 15782 SANTIAGO DE COMPOSTELA

E-mail address: carlos.meninho@gmail.com